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State sum invariants of three-manifolds: A combinatorial approach to topological quantum field theories

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The construction of Turaev and Viro involving quantum 6j-symbols and giving rise to invariants of closed, compact three-manifolds is extended. It leads to invariants of coloured graphs on the boundary of compact three-manifolds. This allows one to derive surgery formulas when cutting along an arbitrary two-manifold. In particular all axioms of a topological quantum field theory may be verified and the dimensions of the associated Hilbert spaces are given by the square of the Verlinde formula.

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Statistics is one of the central concepts in many body quantum systems. Consider a system of two identical particles located at x_1 and x_2 in $\mathbb{R}^d$ with a Schrödinger wave function $\psi(x_1, x_2)$. Under the exchange of the particles with these coordinates Bose or Fermi statistics holds if $\psi(x_2, x_1) = \pm\psi(x_1, x_2)$. Leinaas and Myrheim [LM] were the first to realize that this solution is not the only possible one.

For a quick introduction into the problem consider the classical geometric space–time description shown in fig. 1 of the exchange of the position for two identical particles. This classical picture should reflect itself in two quantum mechanical transformation laws,

$$\psi_+(x_2, x_1) = R\psi(x_1, x_2) ,$$
$$\psi_-(x_2, x_1) = R^{-1}\psi(x_1, x_2) ,$$

with R being the so-called statistics operator. The problem arises whether there are situations with $R \neq R^{-1}$. For relativistic systems the question has been an-

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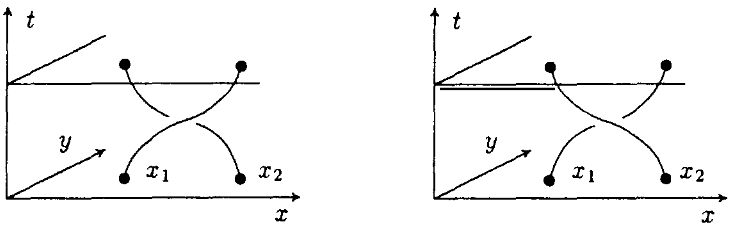


Fig. 1. Two possible ways to exchange two particles.

swered negatively by Doplicher, Haag and Roberts [DHR] for all space dimensions $d \geqslant 3$. Then the statistics is always given by the permutation group. In $d<3$, however, the situation is entirely different and one has the possibility that $R \neq \pm 1$ leading to theories with braid group statistics.

At the moment such theories are not in a very good shape, even the theory of so-called "free particles" is not very well understood. This raises the problem of finding a good starting point for low dimensional quantum field theories with braid group statistics. A structural analysis may be carried out in the context of conformal quantum field theories (see, e.g., refs. [TK, R, RS]) and in algebraic quantum field theory (see, e.g., refs. [F, FRS] and the references quoted therein). A first step in a constructive approach may be provided by topological quantum field theories.

In fact, such theories exhibit no dynamics and even no kinematics but they contain all the structure needed to understand the main effects of braid group statistics. Thus one only has to work within the context of linear algebra in finite dimensional spaces. We briefly review the set-up of topological quantum field theory [At] in its context to statistical physics. Imagine we have a compact three-manifold M (e.g., $M = S^3$, $\Sigma \times S^1$, Σ a compact two-manifold). We imagine there is a partition function given in terms of weights $W(c)$ associated to certain configurations c,

$$Z(M) = \sum_c W(c) . \tag{1}$$

If M is cut along two-surfaces Σ, Σ', Σ'', etc. (see fig. 2), we can write

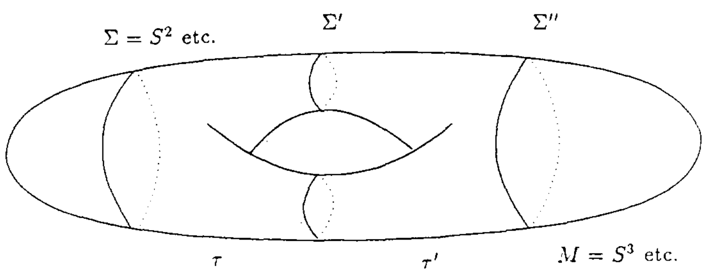


Fig. 2. Cutting a manifold M along Σ, Σ', etc.

$$Z(M) = \text{trace}(\ldots \tau \cdot \tau' \ldots) \,, \tag{2}$$

where the matrix product is taken with respect to vector spaces V^{Σ} such that the transfer matrix τ becomes a linear transformation between such spaces. The typical situation for a topological quantum field theory (i.e. a theory with no dynamics) is that V^{Σ} is finite dimensional and the weights are essentially trivial such that the transfer matrix τ becomes idempotent in the case $\Sigma \cong \Sigma'$. Nevertheless, for nontrivial M and Σ one obtains a nontrivial partition function Z encoded in nontrivial behaviour of τ.

The first example given was the Chern–Simons theory for a fixed gauge group in three dimensions. This was discussed by Witten [W] in terms of a functional integral over the moduli space of all flat connections on the compact oriented manifold M in the form

$$Z_{\text{CS}}(M) = \int \mathscr{D}A \exp \mathrm{i} \int_M \text{tr}(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A) \tag{3}$$

with a vanishing Hamiltonian, i.e. without dynamics. Z_{CS} is then almost trivial such that $V^{\Sigma = S^2}$ is one dimensional. The expectation values of Wilson loops in this theory, on the other hand, give rise to the knot invariants of Jones [J] and braid group representations. Furthermore Riemann surfaces with n punctures Σ_n describe external sources such that the spaces V^{Σ_n} are the associated state spaces for n nonmoving particles obeying braid group statistics. These spaces V^{Σ_n} are analogous to Hilbert spaces associated to internal symmetries (like isospin).

Now there is a model of a topological quantum field theory invented by Turaev and Viro [TV] which is defined combinatorially. Thus one is no longer dependent on difficulties which arise when one tries to give a rigorous definition of functional integrals. This theory is actually the absolute square of the Chern–Simons theory [T]. It is based on the $6j$-symbols of the quantum group $\text{Sl}_q(2, \mathbb{C})$ with $q = \exp \mathrm{i}\pi/r$ ($r \geqslant 3$). Meanwhile this approach has been extended to other quantized versions of classical compact semisimple Lie groups (see, e.g., refs. [DJN1, DJN2]). For definiteness, however, we shall stick to the $\text{Sl}_q(2, \mathbb{C})$ case.

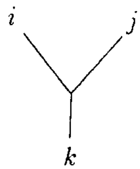
For given r and hence q one restricts attention to representations π_j ($j = 0, \frac{1}{2}, \ldots, \frac{1}{2}r - 1$). Under tensor products one has a decomposition (see, e.g., ref. [RT])

$$\pi_i \otimes \pi_j = \bigoplus_{k : \delta_{ijk} = 1} \pi_k \oplus \text{bad representations} \,, \tag{4}$$

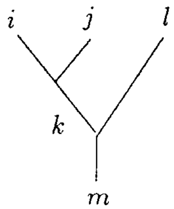
where

$$\delta_{ijk} = \begin{cases} 1 & \text{if } i \leqslant j + k,\, j \leqslant k + i,\, k \leqslant i + j,\, r - 2 \geqslant i + j + k \in \mathbb{Z} \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{5}$$

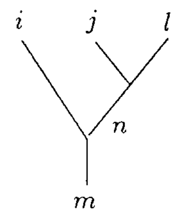
The coupling scheme (4) gives rise to Clebsch–Gordan coefficients, graphically represented by three-vertices,



As in the classical case the quantum $6j$-symbols describe the rearrangement (=associativity) of the coupling



$$= \sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \tag{6}$$



Here

$$w_n^2 = (-1)^{2n}(2n+1)_q , \tag{7}$$

with

$$x_q = \sin(\pi x/r)/\sin(\pi/r) \tag{8}$$

such that $(2n+1)_q$ is the so-called q-dimension of the representation π_n. The weights w_j^2 satisfy the sum rule

$$w_i^2 w_j^2 = \sum_k w_k^2 \delta_{ijk} \tag{9}$$

reflecting relation (4). The classical orthogonality and Biedenharn–Elliot relations persist in the q-deformed case,

$$\sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & j & k' \\ l & m & n \end{vmatrix} = \delta_{ijk}\delta_{lmk} w_k^{-2} \delta_{kk'} , \tag{10}$$

$$\sum_n w_n^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & m & n \\ D & C & A \end{vmatrix} \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} = \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix} . \tag{11c}$$

To the last relation we associate the picture in fig. 3.

In addition, by a convenient choice of phase factors one has the following symmetry relations:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} . \tag{12}$$

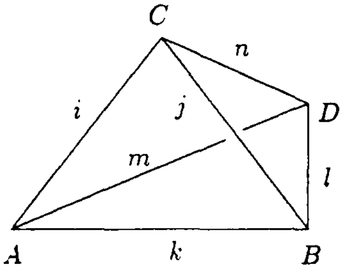


Fig. 3. A tetrahedron representing the Biedenharn–Elliot relation.

The original construction of Turaev and Viro was carried out for the case $\partial M=\emptyset$ and goes as follows. Let X be a triangulation of M with (nonoriented) i-simplices σ^i $(0\leqslant i\leqslant 3)$. For given X define a state sum

$$Z(X)=\sum_{\underline{j}} W(X)(\underline{j}) . \tag{13}$$

Here $\underline{j}$ denotes a configuration in the sense of statistical physics and is given as a colouring of the one-simplices σ^1, i.e. as a map $\sigma^1\mapsto j(\sigma^1)\in\{0, \frac{1}{2}, ..., \frac{1}{2}r-1\}$. For a given configuration the weights $W(X)(\underline{j})$ are given as follows. Let

$$w^2=\sum_j w_j^4 . \tag{14}$$

First we associate elementary weights to the i-simplices $(i\neq 2)$,

$$\sigma^0\in X: w^{-2} ,$$

$$\sigma^1\in X: w_j^2(\sigma^1) ,$$

$$\sigma^3\in X: (6\underline{j})(\sigma^3) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ j(\sigma_4^1) & j(\sigma_5^1) & j(\sigma_6^1) \end{vmatrix} . \tag{15}$$

Here σ_ν^1 $(\nu=1, ..., 6)$ are the one-simplices in the boundary of σ^3 such that σ_ν^1 and $\sigma_{\nu+3}^1$ $(\nu=1, 2, 3)$ are pairwise opposite to each other. Due to the symmetry (12), (15) is a well defined prescription. Now $W(X)(\underline{j})$ is given in terms of these local weight factors as

$$W(X)(\underline{j})= \prod_{\sigma^0\in X} w^{-2} \prod_{\sigma^1\in X} w_{j(\sigma^1)}^2 \prod_{\sigma^3\in X} (6j)(\sigma^3) . \tag{16}$$

The main result in ref. [TV] is now the following.

Theorem [TV]. *$Z(X)$ is independent of the particular choice of the triangulation X of M giving rise to a well defined quantity $Z(M)$ for closed manifolds.*

In collaboration with W. Müller [KMS] we extended the above construction to include the case of manifolds M with nonempty boundary. This generalization also allows for a simplified proof of this theorem and goes as follows. For a given triangulation X of M inducing a triangulation ∂X of ∂M we defined additional configurations $\underline{J}$ as colourings of the vertices σ^0 in ∂X, i.e. $\underline{J}: \sigma^0 \mapsto J(\sigma^0)$ $(\sigma^0 \in \partial X)$. This gives rise to the following additional local weights on the two-simplices σ^2 contained in ∂X,

$$(6\underline{j}, \underline{J})(\sigma^2) = \begin{vmatrix} j(\sigma_1^1) & j(\sigma_2^1) & j(\sigma_3^1) \\ J(\sigma_1^0) & J(\sigma_2^0) & J(\sigma_3^0) \end{vmatrix},$$

where σ_ν^1 $(1 \leqslant \nu \leqslant 3)$ are the one-simplices in $\partial\sigma^2$ and σ_ν^0 $(1 \leqslant \nu \leqslant 3)$ are the zero-simplices in $\partial\sigma^2$ opposite to σ_ν^1.

We then replace $W(X)(\underline{j})$ by

$$W(X)(\underline{j}, \underline{J}) = W(X)(\underline{j}) \prod_{\sigma^2 \in \partial X} (6\underline{j}, \underline{J})(\sigma^2) \cdot \prod_{\sigma^0 \in \partial X} w^2_{J(\sigma^0)} . \tag{17}$$

Accordingly the state sum is now given as

$$Z(X) = \sum_{\underline{j}, \underline{J}} W(X)(\underline{j}, \underline{J}) \tag{18}$$

and we have the following extension of the previous theorem.

Theorem [KMS]. *The state sum $Z(X)$ is independent of the particular choice of the triangulation X of M giving rise to a well defined quantity $Z(M)$ for arbitrary compact three-manifolds M.*

We briefly indicate the proof. First one shows invariance of $Z(X)$ under local isotopies of the boundary ∂X. Thus let $\sigma^3 \in X$ be a three-simplex such that $\partial\sigma^3$ has two two-simplices, say σ_1^1, σ_2^1, in common with ∂X. If we remove σ^3 from X we obtain $X' = X \setminus \sigma^3$ such that σ_1^1 and σ_2^1 are replaced by the two other one-simplices in $\partial\sigma^3$ (see fig. 3). Inspection of (18) in combination with relation (11c) shows that $Z(X) = Z(X')$ holds. In case σ^3 has k one-simplices $(k = 1, 3)$ in common with ∂X one uses the relations

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} = \sum_B w_B^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix} \begin{vmatrix} j & n & l \\ D & B & C \end{vmatrix}, \tag{11b}$$

$$\sum_{\substack{l,m,n \\ D}} w_l^2 w_m^2 w_n^2 w_D^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} \begin{vmatrix} k & m & l \\ D & B & A \end{vmatrix}$$

$$= w^2 \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix}, \tag{11d}$$

which follows from the Biedenharn–Elliot and the orthogonality relations. Invariance under local subdivisions of X (like for example Alexander moves [A]) near ∂X is now easily established. In fact, one first uses local isotopy invariance of the state sum to remove those σ^3 where the local subdivision is supposed to take place. The state sum is then trivially invariant under such local subdivisions. Then one uses again local isotopy invariance in the opposite direction by adding the new σ^3's again. To prove invariance under local subdivisions in the interior of ∂X, we first use the relation [proved analogously to (11b) and (11d)]

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = w^{-2} \sum_{A,B,C,D} w_A^2 w_B^2 w_C^2 w_D^2 \times \begin{vmatrix} i & j & k \\ B & A & C \end{vmatrix} \begin{vmatrix} k & l & m \\ D & A & B \end{vmatrix} \begin{vmatrix} j & n & l \\ D & B & C \end{vmatrix} \begin{vmatrix} i & m & n \\ D & C & A \end{vmatrix} \tag{11a}$$

to remove a three-simplex from the interior creating a hole with boundary $\cong S^2$, changing the state sum by a factor w^{-2}. This hole may be enlarged by the previous argument without changing the state sum. This hole may be chosen so large that the local subdivision takes place within this hole. The previous arguments may be used again such that in the end the hole is filled again, proving the theorem. Relations (11a–d) may be viewed as a local Stokes theorem for the present context. Invariance under subdivisions may be just viewed as renormalization group invariance of this theory, something to be expected of a theory with trivial dynamics.

The relation [established in analogy to (11a, b, d)]

$$\sum_{\substack{i,j,k,l,m,n \\ A,B,C,D}} w_i^2 w_j^2 w_k^2 w_l^2 w_m^2 w_n^2 w_A^2 w_B^2 w_C^2 w_D^2 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & n & m \\ D & A & C \end{vmatrix} \times \begin{vmatrix} j & l & n \\ D & C & B \end{vmatrix} \begin{vmatrix} k & m & l \\ D & B & A \end{vmatrix} \begin{vmatrix} i & k & j \\ B & C & A \end{vmatrix} = w^8 \tag{11e}$$

in particular states that

$$Z(\mathrm{D}^3) = 1 \ ,$$

while (11a) leads to

$$Z(M \setminus \mathrm{D}^3) = w^2 Z(M) \quad (\mathrm{D}^3 \subset \operatorname{int} M) \tag{19}$$

such that in particular

$$Z(\mathrm{S}^3) = 1/w^2 \tag{20}$$

holds.

By similar arguments it is easy to establish that $Z(\mathrm{S}^2 \times \mathrm{S}^1) = 1$ such that with

$$\dim V^{\Sigma} = Z(\Sigma \times \mathrm{S}^1) \tag{21}$$

one has

$$\dim V^{\mathrm{S}2} = 1 \tag{22}$$

for the associated topological quantum field theory. More generally, if M is cut by an S^2 into two disjoint manifolds M_1 and M_2, the relation

$$Z(M)=w^2 Z(M_1)Z(M_2) \tag{23}$$

is valid in accordance with (22).

The question arises what happens if M is cut into two pieces by a general two-manifold Σ. Turaev and Viro suggested the following procedure. For general M with triangulation X inducing a triangulation ∂X of ∂M let

$$Z_{\underline{\alpha}}(X)=\frac{1}{W_{\underline{\alpha}}}\sum_{\underline{j}\mid\partial X=\underline{\alpha}} W(X)(\underline{j}) \tag{24}$$

with

$$W_{\underline{\alpha}}=\prod_{\sigma^1\in\partial X} w^2_{j(\sigma^1)} \,. \tag{25}$$

Then by definition

$$Z(X_1\cup_{\partial X_1=\partial X_2} X_2)=\sum_{\underline{\alpha}} W_{\underline{\alpha}} Z_{\underline{\alpha}}(X_1) Z_{\underline{\alpha}}(X_2) \,. \tag{26}$$

This formulation, however, is triangulation dependent and not well suited for practical purposes.

The following is a triangulation independent formulation with potentially interesting applications. Let M_1 and M_2 be orientable such that $\Sigma\subset\partial M_2$ and $\Sigma^*\subset\partial M_1$. Then the following relation is valid:

$$Z(M_1\cup_\Sigma M_2)=\sum_{\underline{x}} W_{\underline{x}} Z(M_1, G^{\Sigma^*}_{\underline{x}}) G(M_2, G^{\Sigma}_{\underline{x}}) \,. \tag{27}$$

Here $G^{\Sigma}_{\underline{x}}$ is a certain coloured graph, called a canonical graph on Σ with colours $x_i\in\{0, \frac{1}{2}, ..., \frac{1}{2}r-1\}$ associated to maximal intervals l_i on the graph, such that $W_{\underline{x}}=\Pi_i w^2_{x_i}$. $Z(M_2, G^{\Sigma}_{\underline{x}})$ is then a generalization of the state sum $Z(M_2)$, which is invariant under homotopies of the graph on Σ. If, for example, $\Sigma=S^1\times S^1$ then this canonical graph has the form depicted in fig. 4.

For general Σ of genus $g\geqslant$ the graph $G^{\Sigma}_{\underline{x}}$ carries $6g-3$ colours and has the geometric form shown in fig. 5 in the standard polygon description of Σ.

Moreover the matrix

$$\mathbf{1}^{\Sigma}_{\underline{x},\underline{y}}=Z(\Sigma\times I, G^{\Sigma}_{\underline{x}}\cup G^{\Sigma^*}_{\underline{y}}) \tag{28}$$

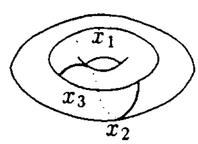


Fig. 4. The canonical graph for the torus $S^1\times S^1$.

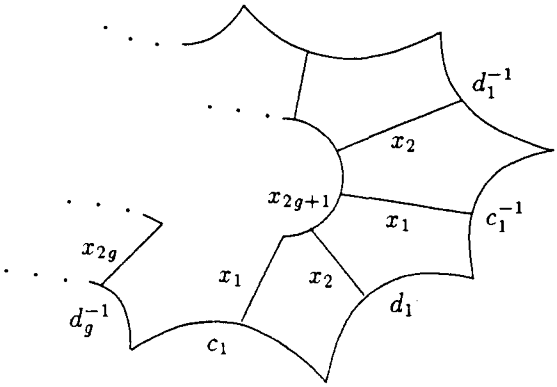


Fig. 5. The canonical graph $|G|_{\underline{x}}^{\Sigma^g}$.

(I is the unit interval) is idempotent in the sense that

$$\mathbf{1}^{\Sigma}_{\underline{x},\underline{y}} = \sum_{\underline{z}} W_{\underline{z}} \mathbf{1}^{\Sigma}_{\underline{x},\underline{z}} \mathbf{1}^{\Sigma}_{\underline{z},\underline{y}} \tag{29}$$

holds. In other words, this is the unit operator in V^{Σ} such that by (27)

$$\dim V^{\Sigma} = \operatorname{tr} \mathbf{1}^{\Sigma} = Z(\Sigma \times S^1) \, . \tag{30}$$

This may be explicitly evaluated in the case that $\Sigma = \Sigma^g$ is connected of genus g and one obtains the square of the Verlinde formula [V],

$$\dim V^{\Sigma^g} = \left[\operatorname{tr} \left(\sum_a N^a N^a \right)^{g-1} \right]^2 , \tag{31}$$

where

$$N^a_{bc} = \delta_{abc} \tag{32}$$

is the fusion matrix associated to (4).

We briefly indicate further generalizations carried out in ref. [KS]. First one may introduce state sums of coloured framed links in M satisfying the usual braiding relations for $SL_q(2, \mathbb{C})$.

Secondly one may define state sums involving what we call left and right handed, coloured punctured Riemann surfaces $\Sigma_{\underline{a},\underline{\bar{b}}}$ [$\underline{a} = (a_1, ..., a_n)$, $\underline{\bar{b}} = (\bar{b}_1, ..., \bar{b}_{\bar{n}})$] such that relation (31) generalizes to

$$\begin{aligned} \dim V^{\Sigma^g_{\underline{a},\underline{\bar{b}}}} &= Z(\Sigma^g_{\underline{a},\underline{\bar{b}}} \times S^1) \\ &= \operatorname{tr} \left[N^{a_1} \cdots N^{a_n} \left(\sum_a N^a N^a \right)^{g-1} \right] \\ &\quad \times \operatorname{tr} \left[N^{\bar{b}_1} \cdots N^{\bar{b}_{\bar{n}}} \left(\sum_a N^a N^a \right)^{g-1} \right] . \end{aligned}$$

Finally we mention that the spaces V^{Σ} and V^{Σ^*} (where Σ^* has orientation opposite to Σ) are dual to each other. Also there is an antilinear map $\tau: V^{\Sigma} \to V^{\Sigma^*}$ such that (i) the induced hermitian scalar product on V^{Σ} is positive definite and (ii) $\tau v(M) = v(M^*)$ for $\partial M = \Sigma$, where $v(M) \in V^{\Sigma}$ is the state corresponding to M. This last relation is the analogue of reflection positivity in euclidean quantum field theories [OS].

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